INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2023

MATHEMATICS: PAPER I

MARKING GUIDELINES

Time: 3 hours 150 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

SECTION A

QUESTION 1

(a) Solve for x:

(i)
$$\frac{1}{3}x^2 = -2x$$
$$x^2 + 6x = 0$$
$$x\left(x+6\right) = 0$$
$$x = 0; x = -6$$

(ii)
$$0,4^{x+2} = 5$$
$$x + 2 = \log_{0,4} 5$$
$$x = -3,76$$

(b) Show, with reasons, that there are no real values of x for which:

$$\log_2\left(5x+1\right) - \log_2\left(x-2\right) = 2$$
$$\log_2\left(\frac{5x+1}{x-2}\right) = 2$$
$$\frac{5x+1}{x-2} = 4$$
$$5x + 1 = 4x - 8$$
$x \neq -9$ Cannot log negative value

(c) Solve simultaneously for x and y:

$$\left(x-3\right)\left(y+1\right) = 18$$
$$x - 4y = 1$$
$$\left(1+4y-3\right)\left(y+1\right) = 18$$
$$\left(4y-2\right)\left(y+1\right) = 18$$
$$\left(2y-1\right)\left(y+1\right) = 9$$
$$2y^2 + y - 10 = 0$$
$$\left(2y+5\right)\left(y-2\right) = 0$$
$$y = \frac{5}{2}\ ;\ y = 2 \qquad x = -9\ ;\ x = 9$$

(d) Solve: $f(x) = f'(x)$ where $f(x) = x^2 + 2x$, giving your answer in surd form.

$x^2 + 2x = 2x + 2$
$x^2 = 2$
$\therefore x = \pm\sqrt{2}$

(e) Write down a possible domain of the function $f(x) = 9 - (x+4)^2$ such that its inverse will be a function.

$\therefore x \geq -4$ *or* $x \leq -4$ *etc*

QUESTION 2

(a) The first term of a geometric sequence is 10 and the sum to infinity is 30. Determine the common ratio.

$30 = \dfrac{10}{1-r}$

$\therefore 1 - r = \dfrac{10}{30}$

$\therefore r = \dfrac{2}{3}$

(b) The sum of the 4th and 8th terms of an arithmetic sequence is 36. The sum of the first 20 terms is 675. Find the first term and common difference.

$a + 3d + a + 7d = 36$
$2a + 10d = 36$
$a + 5d = 18$

$\dfrac{20}{2}\left[2a + 19d\right] = 675$

$4a + 38d = 135$

$4(18 - 5d) + 38d = 135$

$d = 3{,}5$

$a = 0{,}5$

QUESTION 3

(a) Find $f'(x)$ from first principles if $f(x)=\frac{3}{2}x^2-5$

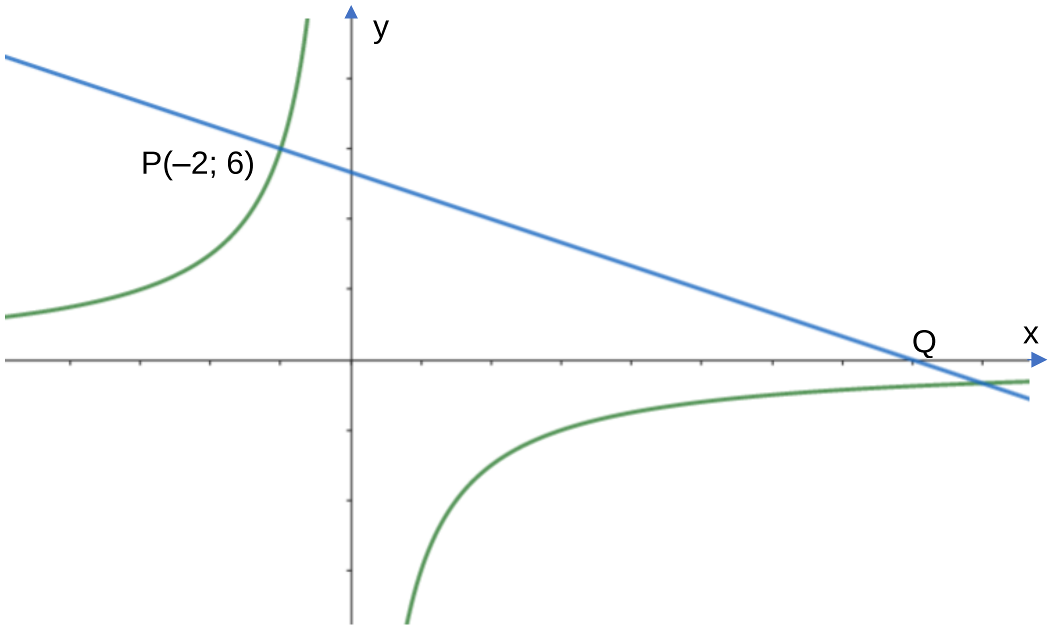
$$f'(x)=\lim_{h\to 0}\frac{\frac{3}{2}(x+h)^2-5-\frac{3}{2}x^2+5}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{3}{2}x^2+3xh+\frac{3}{2}h^2-\frac{3}{2}x}{h}$$

$$f'(x)=\lim_{h\to 0}3x+\frac{3}{2}h$$

$$f'(x)=3x$$

(b) The graph of $f(x)=\frac{a}{x}$ is shown below. Another function, g, is perpendicular to the tangent to f at the point P(–2; 6). The graph of g intersects the x-axis at Q.



(i) Show that a = –12.

$$6=\frac{a}{-2}$$

$$\therefore\ a=-12$$

(ii) Find the gradient of *f* at the point P.

$$f'(x)=12x^{-2}$$

$$f'(-2)=3$$

(iii) Hence, determine the coordinates of Q

$$m = -\frac{1}{3}$$

$$y - 6 = -\frac{1}{3}(x+2)$$

$$0 - 6 = -\frac{1}{3}x - \frac{2}{3}$$

$$-18 = -x - 2$$

$$x = 16 \qquad (16;0)$$

QUESTION 4

(a) An investment of x rand increases to R3 600,77 in 5 years, where the interest rate is 7% per annum compounded monthly. Calculate x, to the nearest rand.

$$3600{,}77 = x\left(1+\frac{0{,}07}{12}\right)^{60}$$

$$x = R2540$$

(b) I invest R5 750 per month, starting immediately, over a period of 8 years. The effective rate is 10% per annum.

(i) Find the nominal interest rate per annum if it is compounded monthly

$$0{,}1 = \left(1+\frac{r}{12}\right)^{12} - 1$$

r = 0,0957

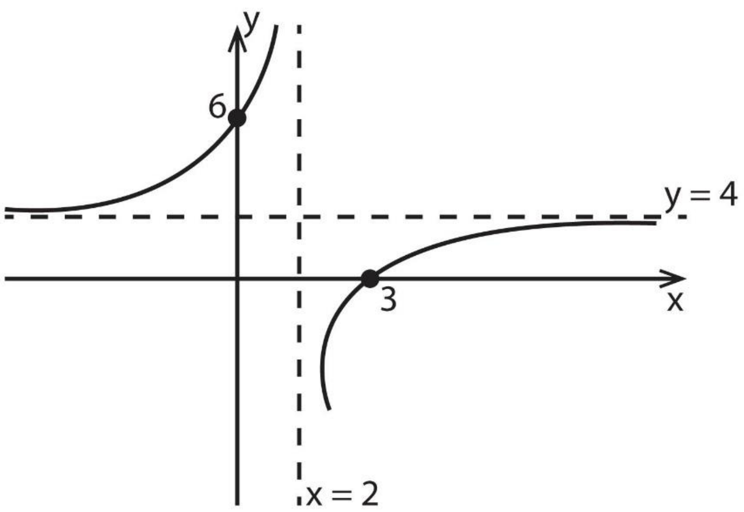
(ii) Calculate the balance in the account after 8 years.

$$F = \frac{5750\left[\left(1+\frac{0{,}0957}{12}\right)^{97} - 1\right]}{\frac{0{,}0957}{12}}$$

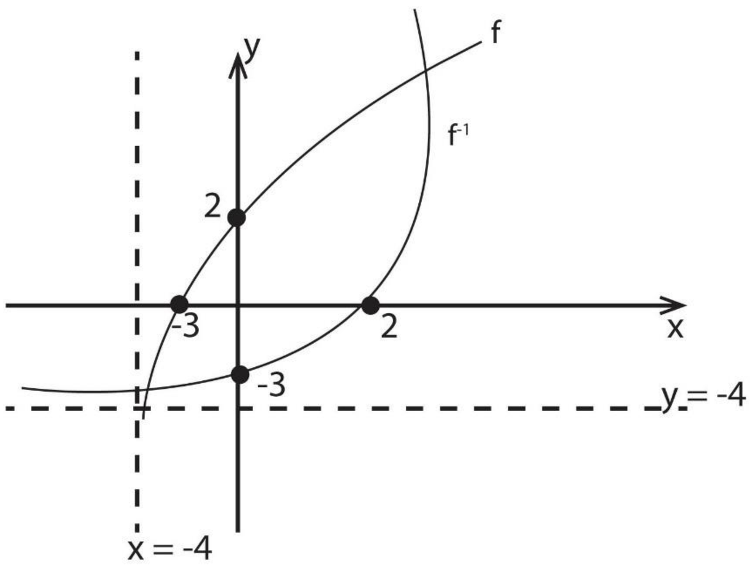
= R836 945,63

QUESTION 5

(a) Sketch the graph of $y = 4 - \dfrac{4}{x-2}$ in the space below:

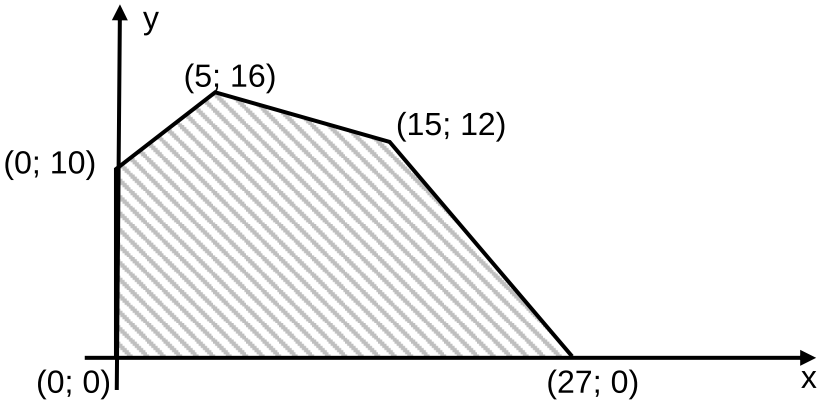


(b) Sketch the graph of $f(x) = \log_2(x+4)$ and $f^{-1}(x)$ on the same axes in the space below.



QUESTION 6

A feasible region is defined by the shaded region in the diagram below. The vertices of the region are as indicated in the diagram.



(a) Write down a set of inequalities which describe the region, R.

$x \geq 0; y \geq 0$

$y \leq \frac{6}{5}x + 10$

$y \leq \frac{-2}{5}x + 18$

$y \leq -x + 27$

(b) Maximise P if $P = 3x + 5y$

$y \leq -\frac{3}{5}x + \frac{p}{5}$

$(15;12)$

$p = 3(15) + 5(12)$

= 105

(c) For which values of k will Q be maximised at (15; 12) if $Q = kx + y$?

$y = -kx + Q$

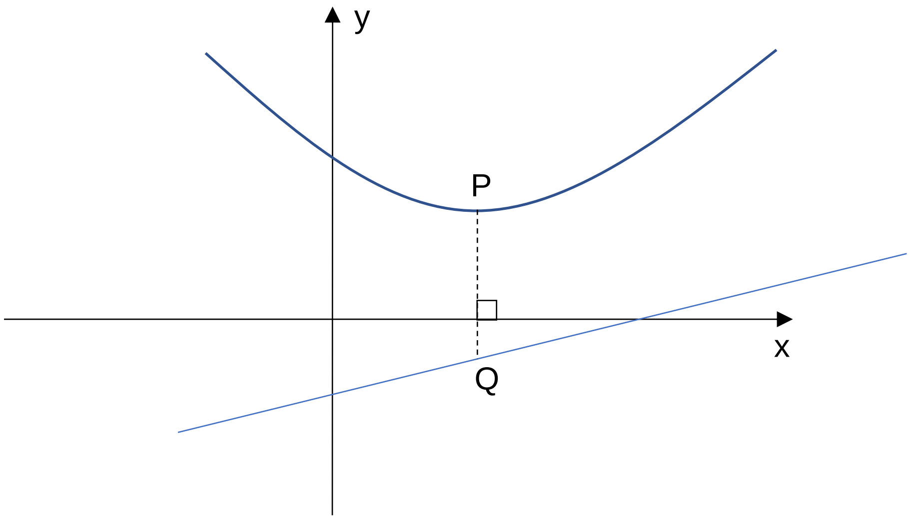
$-1 < -k < -2/5$

$2/5 < k < 1$

SECTION B

QUESTION 7

In the diagram below, $f(x) = x^2 - 6x + 11$ and $g(x) = \frac{1}{2}x - 4$. P is the turning point of f, while Q lies on g. PQ is perpendicular to the x-axis.



(a) Determine the length PQ, if P is the turning point of f.

$P = (3;2)$

$PQ = 2 - \left(-\frac{5}{2}\right)$

$= \frac{9}{2}$

(b) Find the minimum possible length of PQ.

$L(x) = x^2 - 6x + 11 - \frac{1}{2}x + 4$

$= x^2 - \frac{13}{2}x + 15$

$L'(x) = 2x - \frac{13}{2}$

$\therefore x = \frac{13}{4}$ $\quad \therefore L(x) = 4,4375$

QUESTION 8

For which value(s) of k will the equation $\dfrac{4}{x+k} = -6 - x$ have two real, unequal roots?

$4 = (x+k)(-6-x)$

$\therefore 4 = -6x - x - 6k - kx$

$\therefore x^2 + x(6+k) + (4+6k) = 0$

$\therefore (6+k)^2 - 4(1)(4+6k) > 0$

$36 + 12k + k^2 - 16 - 24k > 0$

$\therefore k^2 - 12k + 20 > 0$

$\therefore k < 2 \;\; or \;\; k > 10$

QUESTION 9

(a) The amount of rainfall, per month (n), in a particular year happens to follow a sequence with formula:

$T_n = an^2 + bn + c \quad 1 \le n \le 12$

The rainfall for the first three months is given in the table below:

Month	Jan (n = 1)	Feb (n = 2)	Mar (n = 3)
Rainfall (mm)	68	47	32

(i) Determine the values of a, b and c respectively.

95 68 47 32

–27 –21 –15

6 6

$\therefore a = 3 \quad c = 95 \quad 3 + b + 95 = 68 \quad b = -30$

(ii) Determine the rainfall in December (n = 12).

$T_n = 3n^2 - 30n + 95$

$T_n = 3(12)^2 - 30(12) + 95$

$T_n = 167$ mm

(iii) In which month was the lowest rainfall? Show clear working.

$T^1 = 6n - 30 = 0$

$n = 5$

(b) Solve for p if: $\sum_{k=p}^{4p-1} (2k-1) = 216$

No. of terms $= 4p - 1 - p + 1$

$= 3p$

$3p\left[2(2p-1) + (3p-1)(2)\right] = 216$

$3p(5p-2) = 216$

$15p^2 - 6p - 216 = 0$

$p = 4$

QUESTION 10

The function $f(x) = \frac{2}{9}x^3 + ax^2 + bx$ has a stationary point at (6, 12)

(a) Calculate a and b.

$$12 = \frac{2}{9} \times 6^3 + 36a + 6b$$
$$12 = 48 + 36a + 6b$$
$$\therefore b = -6a - 6$$
$$f'(x) = \frac{2}{3}x^2 + 2ax + b$$
$$O = 24 + 12a + b$$
$$-6a - 6 = -12a - 24$$
$$\therefore a = -3 \qquad \therefore b = 12$$

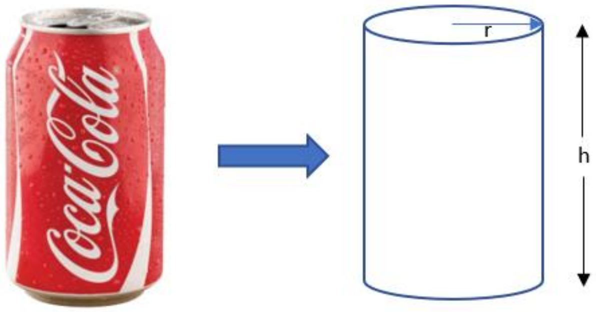
(b) Determine the x-coordinates of the other stationary point and the point of inflection, respectively.

$$f'(x) = \frac{2}{3}x^2 - 6x + 12$$
$$0 = x^2 - 9x + 18$$
$$0 = (x-3)(x-6)$$
$$\therefore x = 3$$
$$f''(x) = \frac{4}{3}x - 6$$
$$x = \frac{9}{2}$$

QUESTION 11

A can of Coke is modelled as the shape of a cylinder, with radius r, height h and a volume of 340 cm^3 (although it is not an exact cylinder in reality).

The cost of making the top and bottom surface of the can is 30 cents/cm^2, whilst the curved surface costs 10 cents/cm^2.



(a) Obtain an expression for h, in terms of r.

$$340 = \pi r^2 h$$
$$\therefore h = \frac{340}{\pi r^2}$$

(b) Hence show that the outer surface area of the can is given by the formula:

$$A = 2\pi r^2 + \frac{680}{r}$$
$$A = 2\pi r^2 + 2\pi r\left(\frac{350}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{680}{r}$$

(c) Now write down an expression for the cost of making a can, and hence determine the radius of the can that will lead to a minimum cost.

$$C = 30\left(2\pi r^2\right) + 10\left(\frac{680}{r}\right)$$
$$= 60\pi r^2 + \frac{6800}{r}$$
$$C' = 120\pi r + \frac{6800}{r^2}$$
$$r^3 = \frac{6800}{120\pi}$$
$$r = 2{,}62\ cm$$

QUESTION 12

A loan of R800 000 is taken over a period of 10 years. Interest is charged at 10% per annum, compounded monthly. The first payment is made one month after the granting of the loan.

(a) Calculate the monthly payment

$$800\,000=\frac{x\left[1-\left(1+\frac{0,1}{12}\right)^{-120}\right]}{\frac{0,1}{12}}$$

$$x=R10572,06$$

(b) Determine the outstanding balance after 5 years.

$$O.B.=800\,000\left(1+\frac{0,1}{12}\right)^{60}-\frac{10572,06\left[\left(1+\frac{0,1}{12}\right)^{-60}-1\right]}{\frac{0,1}{12}}$$

$$x=R497577,77$$

(c) Hence find the interest paid in the 61st month.

$$497577,77\times\frac{0,1}{12}$$

R4146,48

QUESTION 13

(a) Note that the remainder when 2^4 is divided by 5 is 1, since $\frac{16}{5} = 3 \ rem\ 1$.

Determine the remainder when 2^{2023} is divided by 5.

$$\left.\begin{array}{l} 2^1 \to 2 \\ 2^2 \to 4 \\ 2^3 \to 3 \\ 2^4 \to 1 \end{array}\right] repeat$$

$2^5 \to 2$

$2^6 \to 4$

$\therefore$ remainder 2^{2020} is 1

$\therefore$ remainder 2^{2023} is 3

(b) It is given that

$f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...and\ so\ on$

where $-1 < x < 1$.

Calculate $f'\left(\frac{1}{2}\right)$

$f'(x) = 1 + x + x^2 + x^3 + \dots$

$= \frac{1}{1-x}\ if\ -1 < x < 1$

$f'\left(\frac{1}{2}\right) = \frac{1}{1-x} = 2$

Total: 150 marks