INTERNATIONAL SECONDARY CERTIFICATE EXAMINATION
NOVEMBER 2023

FURTHER STUDIES MATHEMATICS (EXTENDED): PAPER II

MARKING GUIDELINES

Time: 1 hour | 100 marks

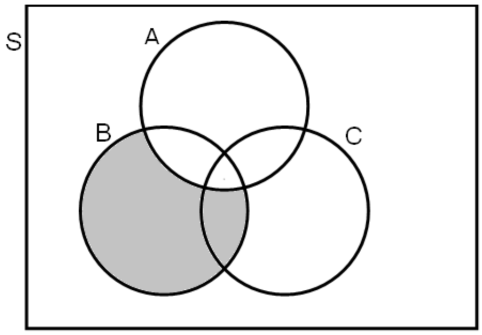
These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

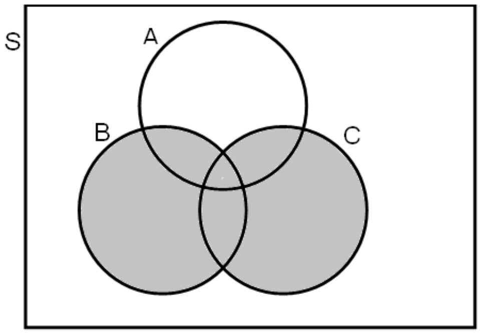
MODULE 2 STATISTICS

QUESTION 1

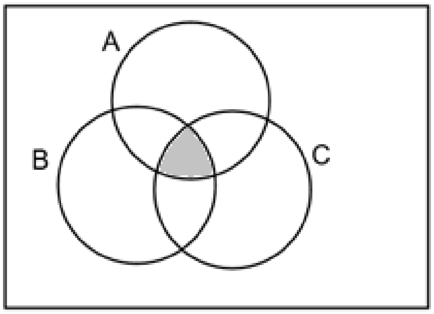
1.1 (a)



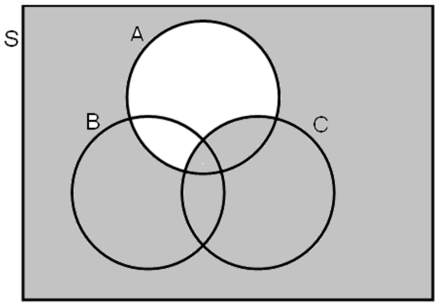
(b)



(c)



(d)



1.2 (a) $P(F \cap L) = \frac{63}{180} = \frac{7}{20}$

(b) $P(F) \times P(L) = P(F \cap L)$

$\frac{108}{180} \times \frac{x}{180} = \frac{7}{20}$

$x = 105$

QUESTION 2

$X \sim N\left(10;4^2\right)$

(a) $$P\left(12 < X < 15\right) = P\left(\frac{12-10}{4} < Z < \frac{15-10}{4}\right)$$
$$= P\left(0,5 < Z < 1,25\right)$$
$$= 0,3944 - 0,1915$$
$$= 0,2029$$

(b) $$P\left(X > a\right) = 0,08$$
$$1,405 = \frac{a-10}{4}$$
$$a = 15,62 \ \therefore 16 \text{ minutes}$$

(c) $$P\left(X > 15 \middle| X > 12\right) = \frac{P\left(X > 15\right)}{P\left(X > 12\right)}$$
$$= \frac{0,5 - 0,3944}{0,5 - 0,1915}$$
$$= 0,3423$$

(d) $$X \sim B\left(6;\ 0,2029\right)$$
$$P\left(X = 2\right) = \binom{6}{2}\left(0,2029\right)^2\left(0,7971\right)^4$$
$$= 0,2493$$

QUESTION 3

3.1 (a) A 96% CI for μ is:

$$\bar{x} = \frac{565}{220} = 2{,}57$$

$$2{,}57 \pm 2{,}05 \times \frac{1{,}45}{\sqrt{220}}$$

$$(2{,}37;\ 2{,}77)$$

(b) 3 is not in the confidence interval hence Naomi's claim is unjustified/ incorrect.

3.2 $H_0 : \mu_x - \mu_y = 140$

$H_1 : \mu_x - \mu_y > 140$

Reject H_0 if $Z > 1{,}645$

Test Statistic:

$$Z = \frac{(1752 - 1598) - 140}{\sqrt{\frac{50^2}{40} + \frac{20^2}{50}}}$$

$$= 1{,}67$$

Conclusion: Since $Z > 1{,}645$ reject the H_0 at the 5% level of significance and hence sufficient evidence to suggest the mean of X is greater than the mean of Y by at least 140.

QUESTION 4

(a) $p = 0,45$

(b) 0,25

(c) $(0,75)^2\,(0,25)$
$= 0,1406$

(d) $E[K] = 1(0,15) + 2\,(0,45) + 3(0,35) = 2,1$
$\sigma^2 = 1(0,15) + 4\,(0,45) + 9(0,35) - (2,1)^2$
$\sigma = 0,8307$

(e) $P(M + K > 4) = P(M = 2)\times P(K = 3) + P(M = 3)\times P(K = 2) + (M = 3)\times P(K = 3)$
$= (0,45)(0,35) + (0,25)(0,45) + (0,25)(0,35)$
$= 0,3575$

QUESTION 5

(a) The mode has the highest probability as it is the value that occurs the most often.

(b) $\int_0^4 kx^2\,dx = \frac{1}{2}$

$$\left[\frac{kx^3}{3}\right]_0^4 = \frac{1}{2}$$

$$\frac{64k}{3} = \frac{1}{2}$$

$$k = \frac{3}{128}$$

(c) $\frac{1}{2}(p-4)\left(\frac{3}{8}\right) = \frac{1}{2}$

$$p - 4 = \frac{8}{3}$$

$$p = \frac{20}{3}$$

QUESTION 6

(a) $\underline{6 \text{ or } 8}$ _ _ _ _ _ _ _

$\frac{7!}{2!2!2!} \times 2 = 1260$ or $\frac{1}{2} \times \frac{8!}{2!2!2!2!}$

(b) 1st option: all 4 numbers are distinct (e.g. 6 842 – has to start with 6 or 8)

$3! \times 2 = 12$ or $\frac{1}{2} \times 4! = 12$

$\left(\text{by symmetry } \frac{1}{2} \text{ of all possible arrangements}\right)$ or $2 \times {}^3P_3 = 12$

2nd option:1 double repeat and 2 distinct (e.g. 6 228 – has to start with 6 or 8)

6_ _ _ + 66_ _ + 8_ _ _ + 88_ _

$$= \left(3 \times 1 \times 2 \times \frac{3!}{2!} + \binom{3}{2} \times 3!\right) \times 2 = 72$$

Or

$$\frac{1}{2} \times \frac{4!}{2!} \times 4 \times \binom{3}{2} = 72$$

$$\left(\begin{array}{l}\frac{1}{2} \text{ of all possible arrangements} \times \text{arrangement} \times \#\text{possible doubles} \\ \times \text{selecting 2 from 3 other numbers}\end{array}\right)$$

Option 3: 2 double repeats (e.g. 6622)

66 _ _ + 88_ _

$$3 \times \frac{3!}{2!} \times 2 = 18$$

Or

$$\frac{1}{2} \times \frac{4!}{2!2!} \times \binom{4}{2} = 18$$

$$\left(\begin{array}{l}\frac{1}{2} \text{ of all possible arrangements} \times \text{arrangement} \times \\ \times \text{selecting 2 pairs from 4 numbers}\end{array}\right)$$

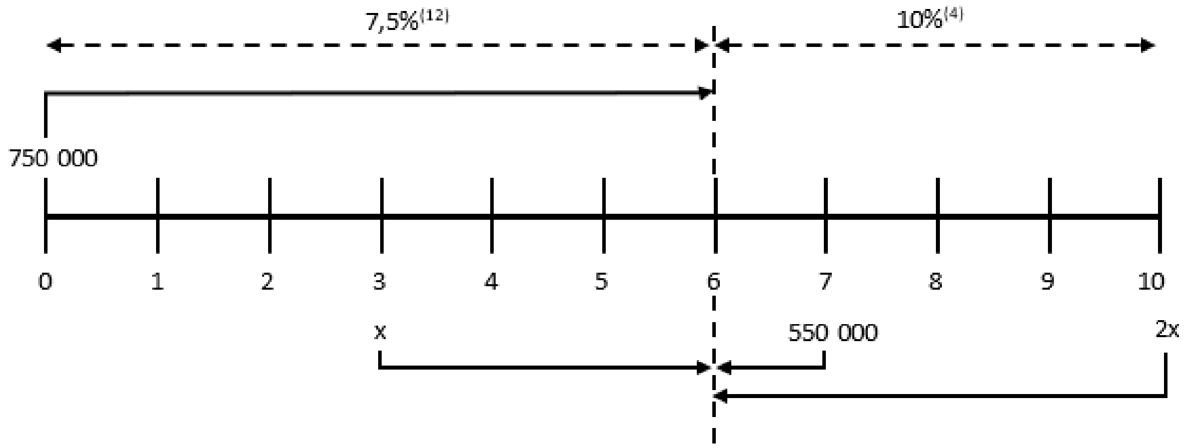
Hence total # options:
$12 + 72 + 18 = 102$

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

(a)



$$750\,000\left(1+\frac{0,075}{12}\right)^{72}=x\left(1+\frac{0,075}{12}\right)^{36}+550\,000\left(1+\frac{0,1}{4}\right)^{-4}+2x\left(1+\frac{0,1}{4}\right)^{-16}$$

$\therefore\ x = \text{R}260251,76$

(b) $550\,000\left(1+\dfrac{0,1}{4}\right)^{-4} + 2\times 260251,76\left(1+\dfrac{0,1}{4}\right)^{-16}$

$= \text{R}\ 848\ 897,01$

[19]

QUESTION 2

$$\left(1+\frac{r}{2}\right)^2 = \left(1+\frac{0,08}{4}\right)^4$$

$$r = 0,0808$$

$$F = \frac{30\,000\left[\left(1+\frac{0,08}{4}\right)^{61}-1\right]}{\frac{0,08}{4}} + \frac{100\,000\left[\left(1+\frac{0,0808}{2}\right)^{30}-1\right]}{\frac{0,0808}{2}}$$

R9166092,92

QUESTION 3

(a) $P \times i = 15\,000$

$$i = \frac{15\,000}{1500\,000} = 0,01$$

$$1\,500\,000\,(1+0,01)^{11} = \frac{x\left[1-(1+0,01)^{-109}\right]}{0,01}$$

$$x = R25\,281,09$$

(b) $$O.B_{72} = 1\,500\,000\,(1+0,01)^{72} - \frac{25\,281,09\left[(1,01)^{61}-1\right]}{0,01}$$

$$= R960\,022,62$$

$$O.B_{84} = 1\,500\,000(1,01)^{84} - \frac{25\,281,09\left[(1,01)^{73}-1\right]}{0,01}$$

$$= R761\,150,02$$

$$\therefore\ \Delta OB = R198\,872,60$$

$$\therefore\ \text{Interest} = 12 \times 25\,781,09 - 198\,872,60$$

$$= 104\,500,48$$

QUESTION 4

(a) (i) 0,08 (2)

(ii) $$m = \frac{0{,}064 - 0{,}08}{400}$$

$$= -0{,}000\,04$$

$$\therefore \frac{r}{k} = 0{,}000\,04$$

$$\therefore k = \frac{0{,}08}{0{,}0000\,4} = 2\,000$$

(iii) $$\frac{\Delta P}{P} = -0{,}000\,04\,P + 0{,}08$$

$$\therefore 0{,}032 = -0{,}000\,04\,P + 0{,}08$$

$$\therefore P = 1\,200$$

(b) $$P_{n+1} = P_n + 0{,}08\,P_n\left(1 - \frac{P_n}{2000}\right)$$

(c) $$\frac{2000 - 200}{2} = 900 \text{ inhabitants}$$

25 – 26 cycles / iterations

QUESTION 5

(a) $a = 0{,}6 \times 1 \times 1 \times 0{,}8$

$= 0{,}48$

(b) $20 \times 10 = 20 \times 500 \times b$

$\therefore\ b = 0{,}02$

(c) $f(0{,}02)(20)(500) = 15$

$\therefore\ f = 0{,}075$

(d) $$95 = \frac{c}{0{,}075 \times 0{,}02}$$

$\therefore\ c = 0{,}1425$

(e) $$20 = \frac{0{,}48}{0{,}02}\left(1 - \frac{0{,}1425}{0{,}015 \times 0{,}02 \times K}\right)$$

$K = 570$

QUESTION 6

(a) $9 = 5a - 3 + b$

$\therefore b = 12 - 5a$

$15 = 9a - 5 + b$

$b = 20 - 9a$

$\therefore a = 2$ and $b = 2$

(b) $T_5 = 2(15) - 9 + 2$

$= 23$

$T_6 = 2(23) - 15 + 2$

$= 33$

(c) Difference increases by 2 …

$\therefore T_n = T_n - 1 + 2(n - 1)$

$\therefore T_n = T_{n-1} + 2n - 2$

Total for Module 3: 100 marks

MODULE 4 GRAPH THEORY & MATRICES

QUESTION 1

1.2 Two columns are a repeat of each other, or volume is zero

1.3 (a) $A^T = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 2 & 4 \\ 0 & -p & 0 \end{bmatrix}$

(b) $\det(A) = 0 - (-p)(4+6) + 0$ using column 3
$= 10p$

$\det(A^T) = 0 - (-p)(4+6) + 0$ using row 3
$= 10p$
$\therefore \det(A) = \det(A^T)$

(c) $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & -7 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ 3 & 2 & 4 \\ 0 & -7 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 10 \\ 5 & 54 & 10 \\ 10 & 10 & 20 \end{pmatrix}$ for each row (column)

(d) Yes, as A is a reflection of A^T

QUESTION 2

$$x+3y-2z=-7$$
$$4x+y+z=5$$
$$2x-5y+7z=19$$

$$\begin{bmatrix}1 & 3 & -2\\ 4 & 1 & 1\\ 2 & -5 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-7\\ 5\\ 19\end{bmatrix}$$

$R_2-4R_1\ ;\ R_3-2R_1$

$$\begin{bmatrix}1 & 3 & -2 & : & -7\\ 0 & -11 & 9 & : & 33\\ 0 & -11 & 11 & : & 33\end{bmatrix}$$

R_3-R_2

$$\begin{bmatrix}1 & 3 & -2 & : & -7\\ 0 & -11 & 9 & : & 33\\ 0 & 0 & 2 & : & 0\end{bmatrix}$$

$z=0$

$-11y = 33$
$y = -3$

$x + 3(-3) + 0 = -7$
$x = 2$

QUESTION 3

3.1 $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 4 & -1 \\ 2 & 2 & 4 & 4 \end{pmatrix}$

3.2 Area is 2 x the original.

3.3 $+\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$

3.4 $m = \dfrac{2}{3}$

tan $\theta = {}^2\!/_3$ Rad = 0,588

$\theta = 33{,}69$

$\begin{bmatrix} \cos 33{,}69 & -\sin 33{,}69 \\ \sin 33{,}69 & \cos 33{,}69 \end{bmatrix}\begin{bmatrix} -1 & 4 & 4 & -1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ pre-multiplying

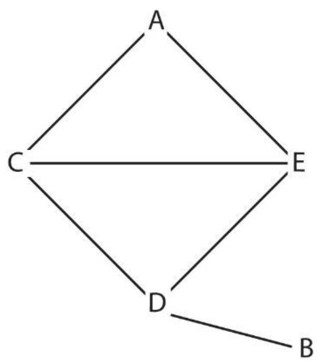
$\begin{pmatrix} -1{,}387 & 2{,}774 & 2{,}219 & -1{,}941 \\ 0{,}277 & 3{,}051 & 3{,}883 & 1{,}109 \end{pmatrix}$

QUESTION 4

4.1 (a)

	A	**B**	**C**	**D**	**E**
A	–		1		1
B		–		1	
C	1		–	1	1
D		1	1	–	1
E	1		1	1	–

(b)



Degree
Edges

(c) 4 points have odd degree

4.2 Forced to form a circuit

QUESTION 5

5.1 E – G 75 (compulsory)

G – D	30 (or ED)
G – F	35
F – A	40
E – B	45
G – C	65
	215

Cable length 290

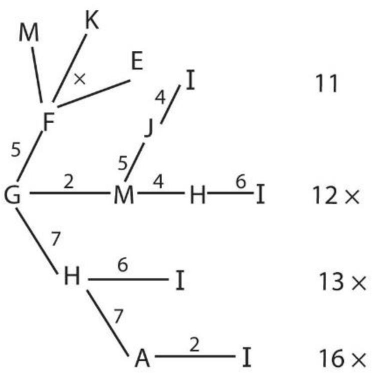
5.2 (a) Sum of edges = 111

Isolate odd degree edges A; C; E; G

Add in: CBA 10
EFG 10

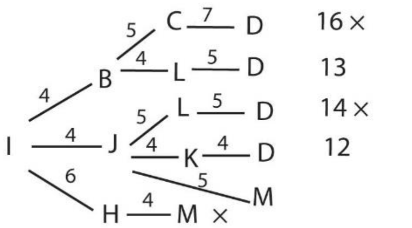
$\therefore$ shortest = 111 + 10 + 10 = 131

5.2 (b) Tourist $\longrightarrow$ information



Information to toilet

G $\longrightarrow$ M – J – I 11 units



I $\rightarrow$ J – K – D 12 units
$\therefore$ 23 units

QUESTION 6

6.1 $$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$
$$= \begin{pmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$
$\therefore$ B is nilpotent

6.2 (a) $P = 2$

(b) $$I - A = \begin{pmatrix} -1 & +2 & 4 \\ +1 & -2 & -4 \\ -1 & +2 & 4 \end{pmatrix}$$

$$(I - A)^2 = \begin{pmatrix} -1 & +2 & 4 \\ +1 & -2 & -4 \\ -1 & +2 & 4 \end{pmatrix} = I - A$$

OR

$$(I - A)^2 = I^2 - 2IA + A^2$$
$$= I - 2A + A \text{ (since } A^2 = A\text{)}$$
$$= I - A$$

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Total for Module 4: 100 marks